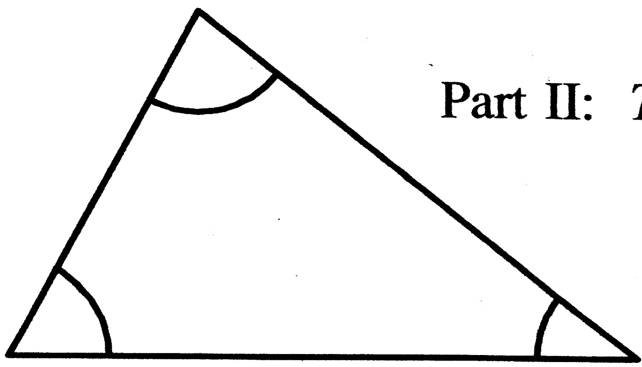
Project MATHEMATICS!

Program Guide and Workbook

to accompany the videotape on

SINES AND COSINES



Part II: *Trigonometry*

Project MATHEMATICS!
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, CA 91125 • (818) 395-3759 or (818) 356-3759

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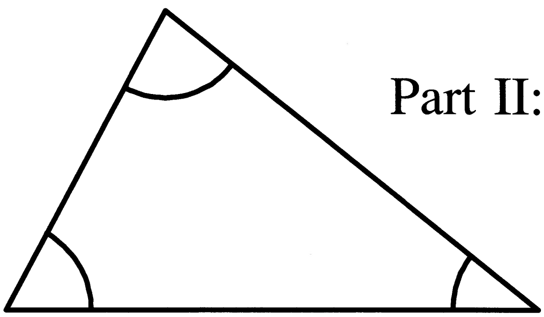

Part II: *Trigonometry*

Written by TOM M. APOSTOL, California Institute of Technology

with the assistance of the

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The videotape on

SINES AND COSINES II

was produced by Project MATHEMATICS!

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AIMS AND GOALS OF *Project MATHEMATICS!*

Project MATHEMATICS! produces computer-animated videotapes to show students that learning mathematics can be exciting and intellectually rewarding. The videotapes treat mathematical concepts in ways that cannot be done at the chalkboard or in a textbook. They provide an audiovisual resource to be used together with textbooks and classroom instruction. Each videotape is accompanied by a workbook designed to help instructors integrate the videotape with traditional classroom activities. Video makes it possible to transmit a large amount of information in a relatively short time. Consequently, it is not expected that all students will understand and absorb all the information in one viewing. The viewer is encouraged to take advantage of video technology that makes it possible to stop the tape and repeat portions as needed.

The manner in which the videotape is used in the classroom will depend on the ability and background of the students and on the extent of teacher involvement. Some students will be able to watch the tape and learn much of the material without the help of an instructor. However, most students cannot learn mathematics by simply watching television any more than they can by simply listening to a classroom lecture or reading a textbook. For them, interaction with a teacher is essential to learning. The videotapes and workbooks are designed to stimulate discussion and encourage such interaction.

STRUCTURE OF THE WORKBOOK

The workbook begins with a brief outline of the video program, followed by suggestions of what the teacher can do before showing the tape. Numbered sections of the workbook correspond to capsule subdivisions in the tape. Each section summarizes the important points in the capsule. Some sections contain exercises that can be used to strengthen understanding. The exercises emphasize key ideas, words and phrases, as well as applications. Some sections suggest projects that students can do for themselves.

I. BRIEF OUTLINE OF THE PROGRAM

The videotape begins with a brief *Review of Sines and Cosines, Part I,* dealing with ideas introduced in a previous program. It explains that sines and cosines show up in many different contexts. For example, if a point traces out an angle of t radians on a unit circle with center at the origin, the horizontal and vertical coordinates of the point are $\cos t$ and $\sin t$. In this context, the sine and cosine are known as *circular functions.*

Graphs related to musical sounds or other periodic phenomena can be analyzed with the help of sines and cosines. Sines and cosines also occur as ratios of lengths of sides of right triangles. Expanding or contracting a right triangle by a scaling factor changes the lengths of its sides, but does not change the angles or the ratios of lengths of corresponding sides. If t denotes the measure of an acute angle in a right triangle, then the sine of t is equal to the length of the side opposite the angle divided by the hypotenuse, and the cosine of t is the length of the side adjacent to the angle divided by the hypotenuse. In this context, the sine and cosine are called *trigonometric functions.*

The sine and cosine functions enjoy many properties, some of which were discussed in the program *Sines and Cosines, Part I.* For example, both functions are periodic with period 2π:

$$\sin(x + 2\pi) = \sin x, \quad \text{and} \quad \cos(x + 2\pi) = \cos x.$$

Further properties of sines and cosines are expressed by the equations

$$\sin(\pi - x) = \sin x, \qquad \sin(\pi + x) = -\sin x.$$

$$\cos x = \sin(\tfrac{\pi}{2} - x), \qquad \sin x = \cos(\tfrac{\pi}{2} - x).$$

This program develops two additional properties of importance in trigonometry: the *law of cosines,* which relates the lengths of the three sides and one angle in any triangle, and the *law of sines*, which states that in any triangle the sine of an angle divided by the length of the opposite side is constant. Applications include a description of the method used to survey the Indian subcontinent and to determine the height of Mt. Everest, the world's tallest peak.

Later programs describe further properties of sines and cosines, including *addition formulas* for determining the sine and cosine of a sum of two numbers, area and slope properties of the graphs of the sine and cosine functions, and the use of polynomial approximations for calculating sines and cosines to any desired degree of accuracy.

II. BEFORE WATCHING THE VIDEOTAPE

This videotape builds on ideas introduced in two earlier modules, *Similarity,* and *Sines and Cosines, Part I.* The ideas are listed below and are discussed briefly in *Review of Sines and Cosines, Part I.* If students are familiar with these ideas, this section will serve as a review. If not, an effort should be made to acquaint them with these ideas and with the key words and statements listed below before viewing the rest of the tape. A good way to do this is to have them read the section entitled *Review of Sines and Cosines, Part I,* and solve some of the exercises in the workbook for part I; specifically, those on p. 11 for radian measure, and on pp. 14 and 26 for special values of the sine and cosine.

KEY WORDS AND STATEMENTS:

Expanding or contracting a plane figure by a *scaling factor* produces a *similar* figure, that is, a figure of the same shape but possibly of different size.

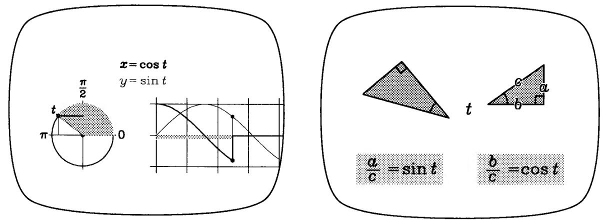
A *radian* is that angle which, when placed at the center of a circle, subtends an arc equal in length to the radius. A right angle contains $\pi/2$ radians, a straight angle contains π radians.

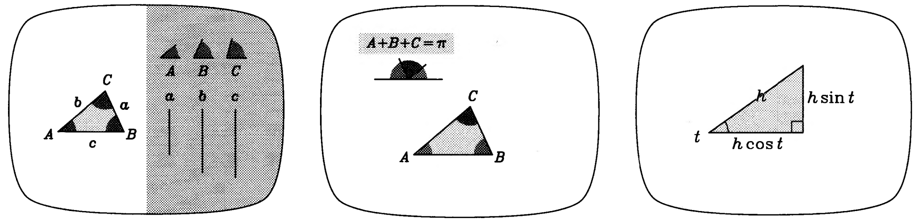
THE MAIN IDEAS:

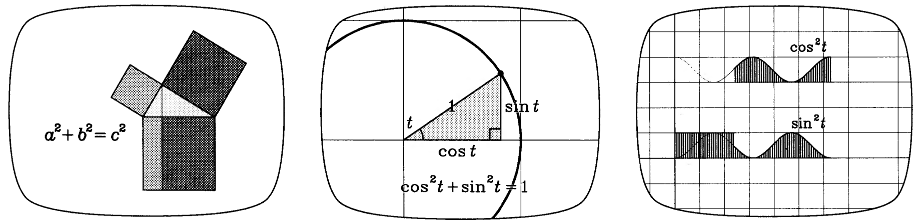
From Similarity: Expanding or contracting a plane figure by a scaling factor multiplies all distances by the same factor. Corresponding angles are equal, lengths of corresponding sides have the same ratio, and corresponding internal ratios are equal. If a figure is scaled by a factor s, lengths of line segments and perimeters are multiplied by s, and surface areas are multiplied by s^2.

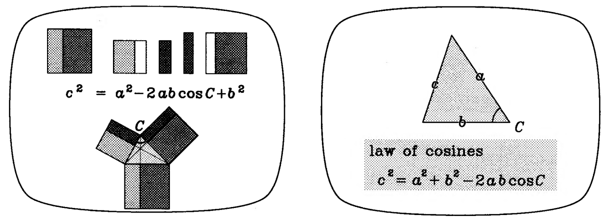
From Sines and Cosines, Part I: Sines and cosines appear in many different contexts. They occur as the rectangular coordinates of a point moving on a unit circle, and as ratios of sides of a right triangle. They are used to analyze graphs related to musical sounds or other periodic phenomena. They play a fundamental role in trigonometry and also help us better understand oscillating systems.

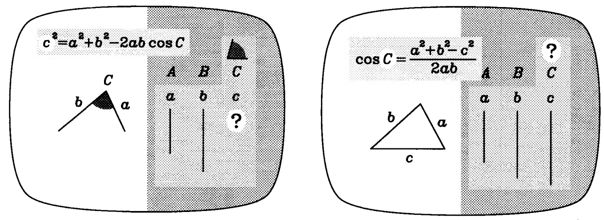
III. TABLE OF CONTENTS



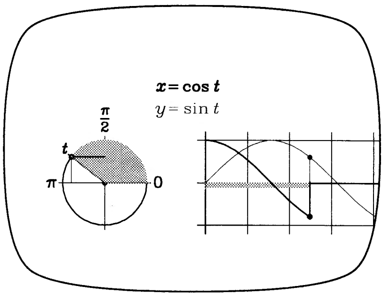


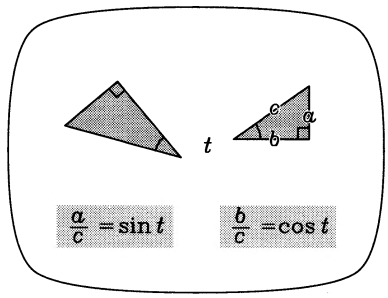






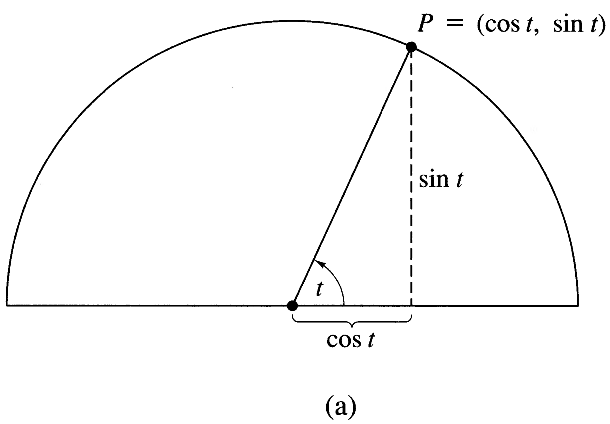
Review of Sines and Cosines, Part I





Ideas from Similarity: Expanding or contracting a plane figure by a scaling factor multiplies all distances by the same factor. Corresponding angles are equal, lengths of corresponding sides have the same ratio, and corresponding internal ratios are equal.

Ideas from Sines and Cosines, Part I: Sines and cosines show up in many different ways. For example, as the rectangular coordinates of a point moving around a unit circle (Figure 1a); as graphs related to musical sounds or other periodic phenomena; or as ratios of lengths of sides of right triangles. Expanding or contracting a right triangle changes the lengths of its sides, but does not change its angles or the ratios of lengths of corresponding sides. The sine of an angle in a right triangle is the length of the opposite side divided by the length of the hypotenuse, and its cosine is the length of the adjacent side divided by the length of the hypotenuse (Figure 1b).



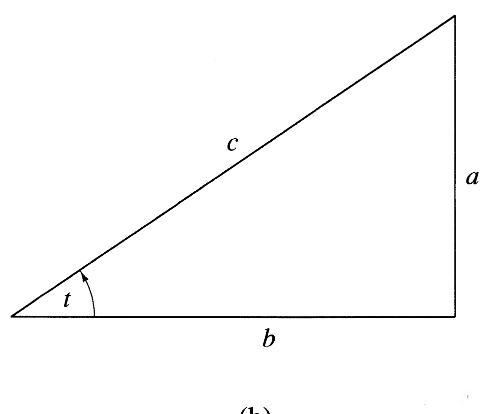
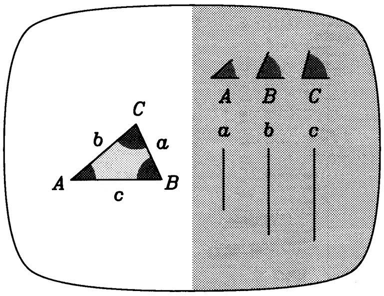
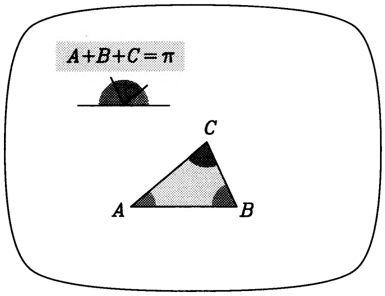
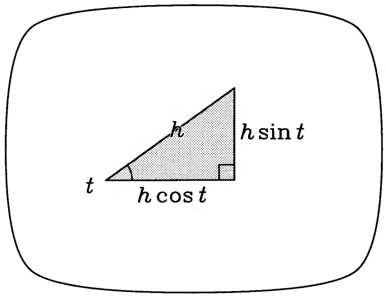


Figure 1. (a) The rectangular coordinates of point P on a unit circle are $(\cos t,\ \sin t)$.
(b) Sine and cosine as ratios of lengths of sides in a right triangle: $\sin t = a/c$, $\cos t = b/c$.

1. *Trigonometry*







The word *trigonometry* is a combination of two Greek words: *trigonon*, meaning triangle, and *metron*, to measure. The word appeared in print in the late 16th century when it was used as the title of an exposition by Bartholomaeus Pitiscus (1561-1613) first published in 1595 as a supplement to a book on spheres. The Greek word for angle is *gonia*, and earlier writers spoke of *goniometry* as the science of angle measurement.

Trigonometry studies relations between sides and angles of triangles. One of its principal uses is to determine distances that are difficult or impossible to measure directly: the height of a tall tree or a pyramid, the width of a deep gorge, the length of a proposed tunnel, or the distance from the earth to the moon. Problems of this type occur in surveying, navigation, large-scale construction, or astronomy. Some are solved by introducing a triangle with the required distance as one of its sides, the triangle being chosen so that enough of its remaining sides or angles can be measured to deduce the required distance.

In the 6th century B.C. the Greek mathematician Thales used properties of similar triangles to determine the height of a column by comparing the length of its shadow with that of his staff. The method (described in the program *Similarity*) is to measure shadows at the time of day when the staff's shadow is equal to the height of the staff; then the column's shadow is equal to the height of the column. But the sun doesn't always shine, or perhaps we cannot wait for a particular time of day to measure shadows. Trigonometry removes these constraints and provides general techniques for making indirect measurements, all based on properties of similar figures.

A triangle has six parts: three angles and three sides. If two angles are known we can find the third because the sum of the angles in a triangle is a straight angle. Expanding or contracting a triangle does not change its angles, so if only the angles are known we cannot determine any of the sides. But if one side and two other parts are known, the remaining parts can be found using two properties of triangles that are the main focus of this program: *the law of cosines* and *the law of sines.*

The law of cosines tells us how to find the length of one side of a triangle in terms of the lengths of the other two sides and the cosine of the angle between them. The law of sines states that in any triangle the ratio of the length of a side to the sine of the opposite angle is constant. This program describes these properties and some of their applications.

The Pythagorean theorem, which is a special case of the law of cosines, was treated in an earlier program. Its connection with sines and cosines is discussed next.

2. *Sines, cosines and the Pythagorean Theorem*

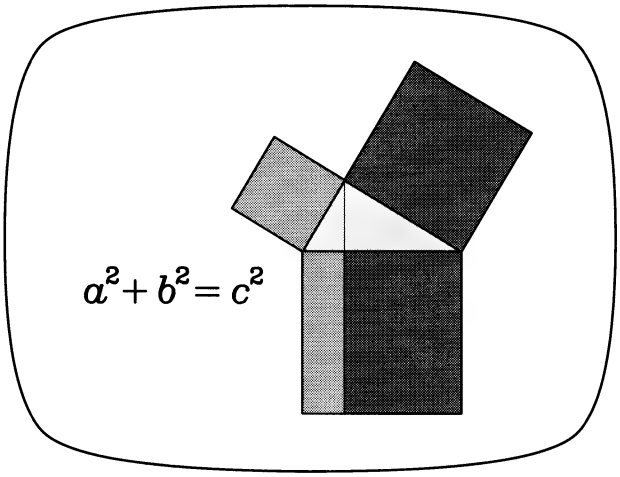


Figure 2. The square on the hypotenuse has area equal to the sum of the areas of the squares on the legs.

According to the Pythagorean Theorem, if a square is drawn on each side of a right triangle, as in Figure 2, the square on the hypotenuse has area equal to the sum of the areas of the squares on the legs.

In a right triangle with base angle t and hypotenuse 1 the legs have length $\sin t$ and $\cos t$, as shown in Figure 3, so the Pythagorean Theorem yields the identity

$$\cos^2 t + \sin^2 t = 1,$$

which holds for every t. This is another fundamental relation connecting the sine and cosine functions.

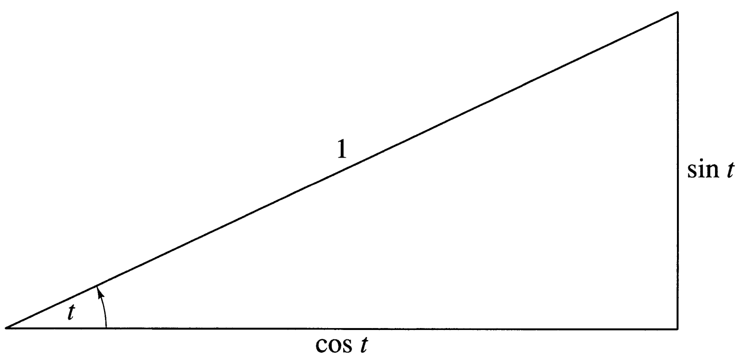


Figure 3. A right triangle with hypotenuse 1 and legs of length $\sin t$, $\cos t$.

The *Theorem of Pythagoras* videotape contains an animated version of Euclid's proof of the Pythagorean theorem. Draw a perpendicular from the right angle to the opposite side, and extend it to divide the large square into two rectangles. By a combination of shearing and rotation, a blue square erected on one leg of the right triangle is transformed into a blue rectangle erected on part of the hypotenuse. (Figure 4.) The same process is repeated, transforming a green square on the other leg to a green rectangle erected on the rest of the hypotenuse. The blue and green rectangles together fill out a square on the hypotenuse, and the proof of the theorem is revealed visually.

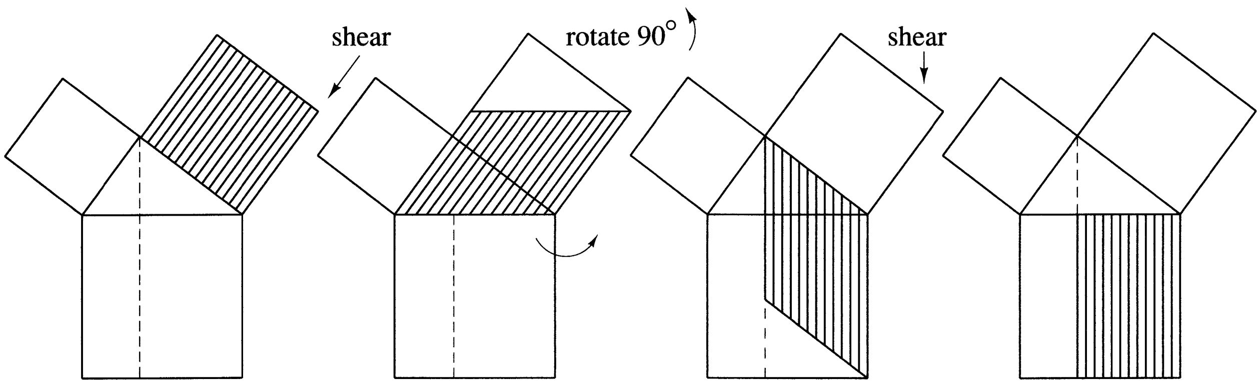


Figure 4. Animated version of Euclid's proof of the Pythagorean Theorem.

In a variation of the foregoing proof, the blue square on one leg is sheared into a parallelogram of the same area, and the parallelogram is then sheared in another direction to become the blue rectangle on the hypotenuse. The same is done with the green square on the other leg thus revealing again that the area of the square on the hypotenuse is the sum of the areas of the squares on the two legs. (Figure 5.)

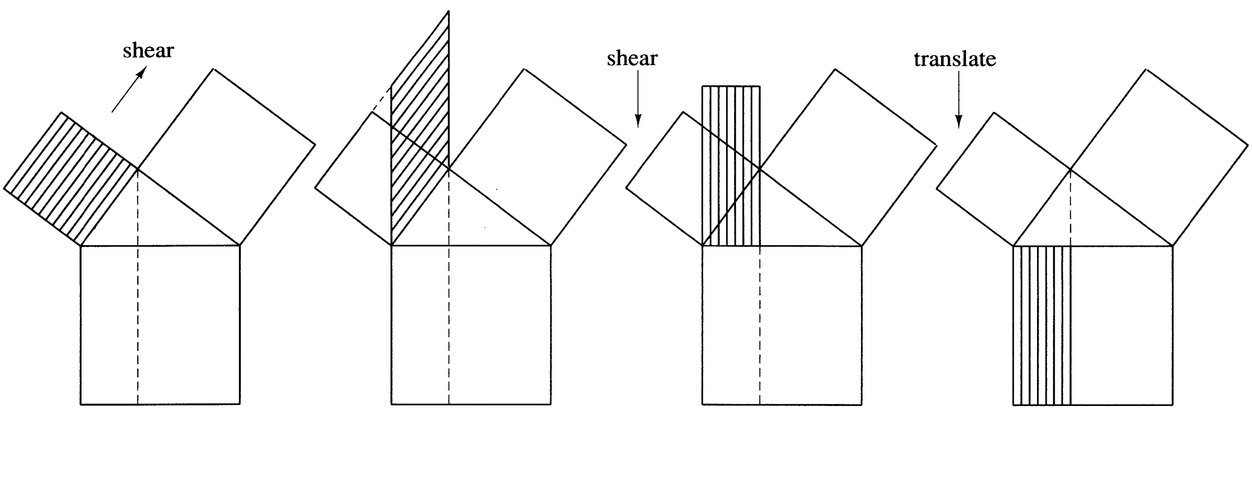


Figure 5. Variation of the animated proof of the Pythagorean Theorem using shearing but no rotation.

The methods in these animated proofs can also be used when the right angle is replaced by a larger or smaller angle. For example, if the right angle is decreased to a smaller angle and we apply the variation using shearing but no rotation, we discover that the sum of the squares constructed on two sides of the triangle is larger than the area of the square on the third side. (Figure 6.) How much larger is it? The answer is revealed by the *law of cosines*, described in the next section.

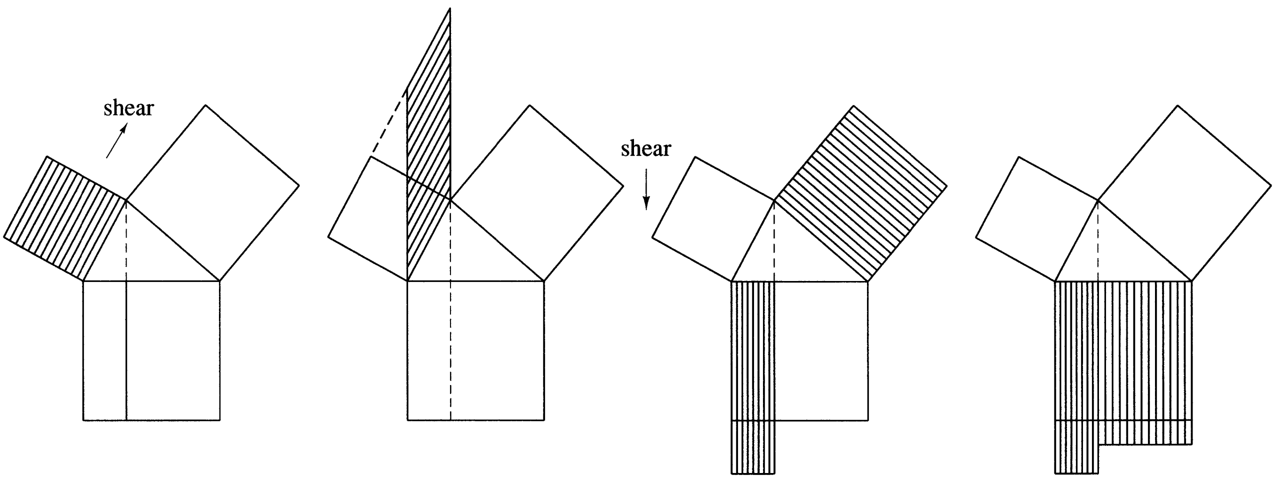


Figure 6. A triangle with the sum of the squares of two sides larger than the square of the third side.

If, on the other hand, the right angle is increased to a larger angle, the sum of the squares on the two sides is smaller than the square on the third side. (Figure 7.) Again, the exact discrepancy is explained by the law of cosines.

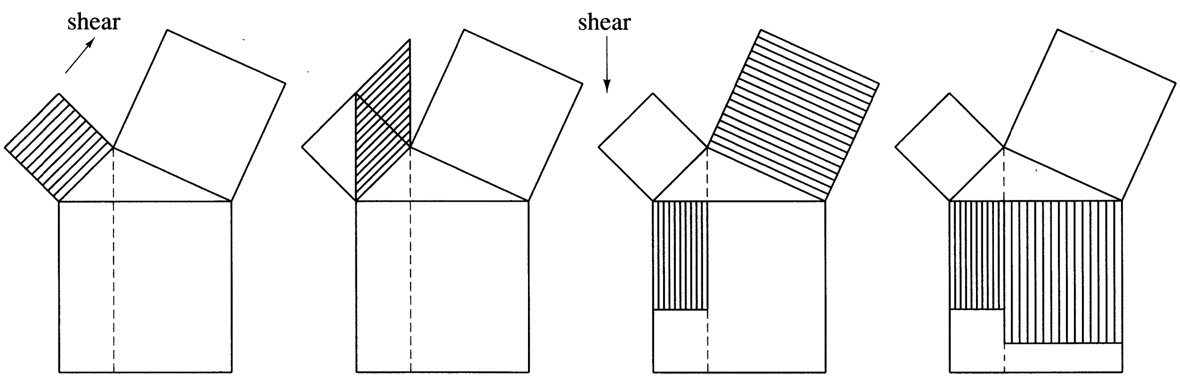


Figure 7. A triangle with the sum of the squares of two sides smaller than the square of the third side.

3. *The law of cosines*

In Figure 8, each triangle ABC has edges of length a, b, c opposite the angles A, B, C, respectively. The law of cosines relates the lengths of the edges by the equation

$$c^2 = a^2 + b^2 - 2ab \cos C. \qquad \textbf{(law of cosines)}$$

In words, the square of one side of a triangle equals the sum of the squares of the other two sides minus twice the product of the lengths of these sides multiplied by the cosine of the angle between them. When C is a right angle, as in Figure 8a, $\cos C = 0$ and the law of cosines reduces to the Theorem of Pythagoras. In Figure 8b, C is smaller than a right angle, $\cos C$ is positive, and c^2 is less than $a^2 + b^2$. And in Figure 8c, C is larger than a right angle, $\cos C$ is negative and c^2 is larger than $a^2 + b^2$. This section gives different proofs of the law of cosines.

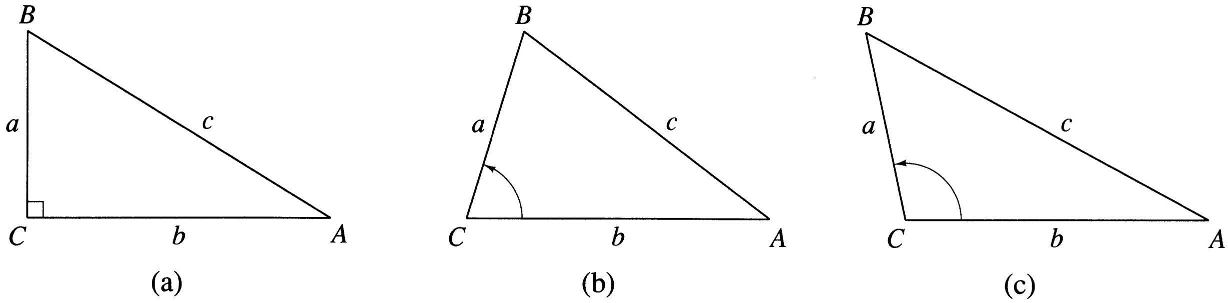


Figure 8. The law of cosines (an extension of the Pythagorean theorem): $c^2 = a^2 + b^2 - 2ab \cos C$.

An argument using animation

Figure 9a shows a triangle ABC with edges of length a, b, c opposite angles A, B, C. In this example, angle C is smaller than a right angle. A square is drawn on each edge, and the altitude from each vertex of the triangle is extended to dissect its opposite square into two rectangles as shown in Figure 9a. The

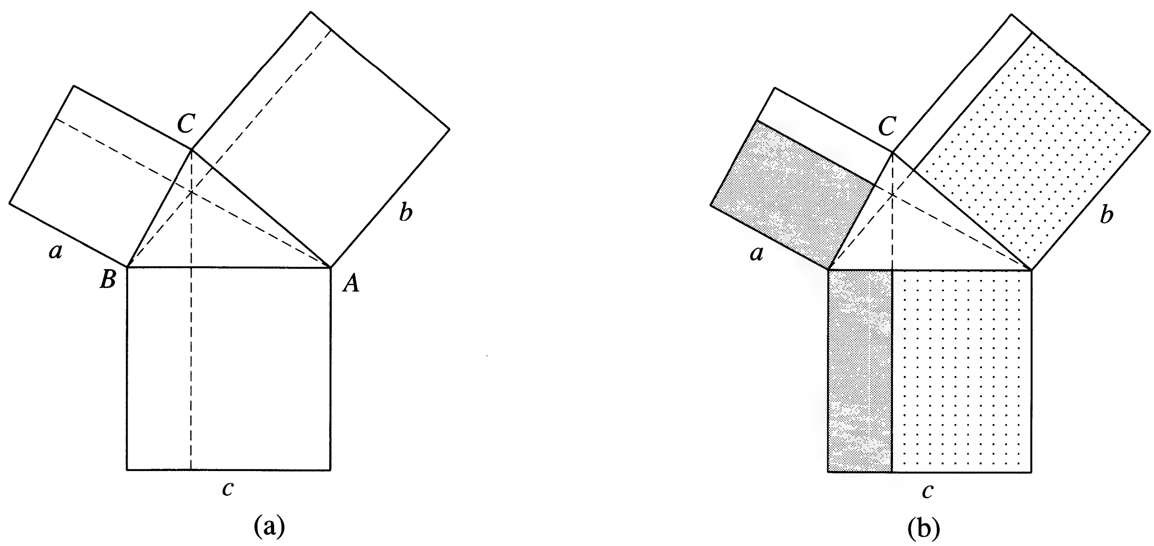


Figure 9. (a) Squares divided into rectangles. (b) Correspondingly shaded rectangles have equal area.

correspondingly shaded rectangles in Figure 9b have equal areas, as suggested by Figure 10, which shows a shearing-rotation-shearing argument similar to that used to prove the Pythagorean Theorem.

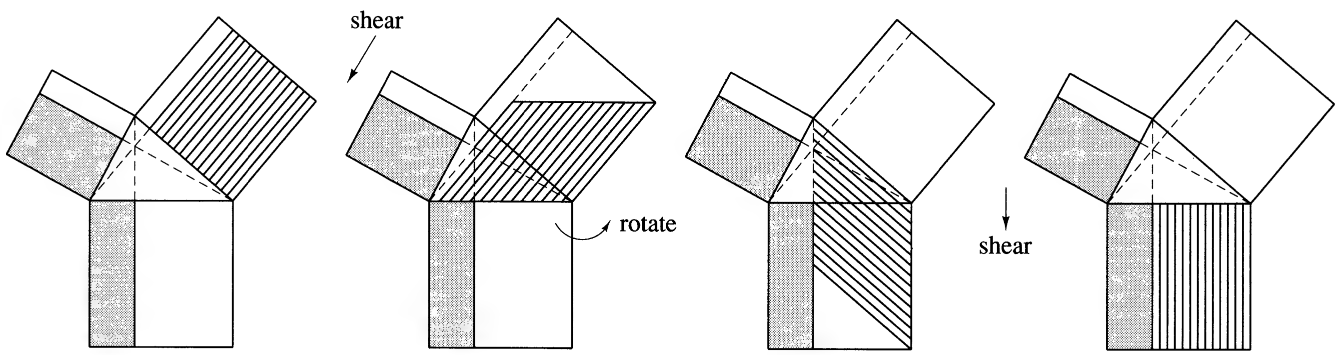


Figure 10. Areas of the shaded regions are unchanged by shearing and rotation.

We can also show that the correspondingly shaded rectangles have equal areas by direct calculation. In Figure 11 the unshaded rectangle inside the square of side a has one edge of length $b \cos C$, so its area is the product $a(b \cos C)$. The adjacent shaded rectangle inside the square of side a has area equal to the difference $a^2 - ab \cos C$. This shaded rectangle can be sheared into a parallelogram of the same area with one edge along the square of side c, then rotated by 90° and sheared again to form the shaded rectangle inside the square of side c. In the same way, the dotted rectangle inside the square of side b can be sheared, then rotated and sheared again to form the dotted rectangle inside the square of side c. Each

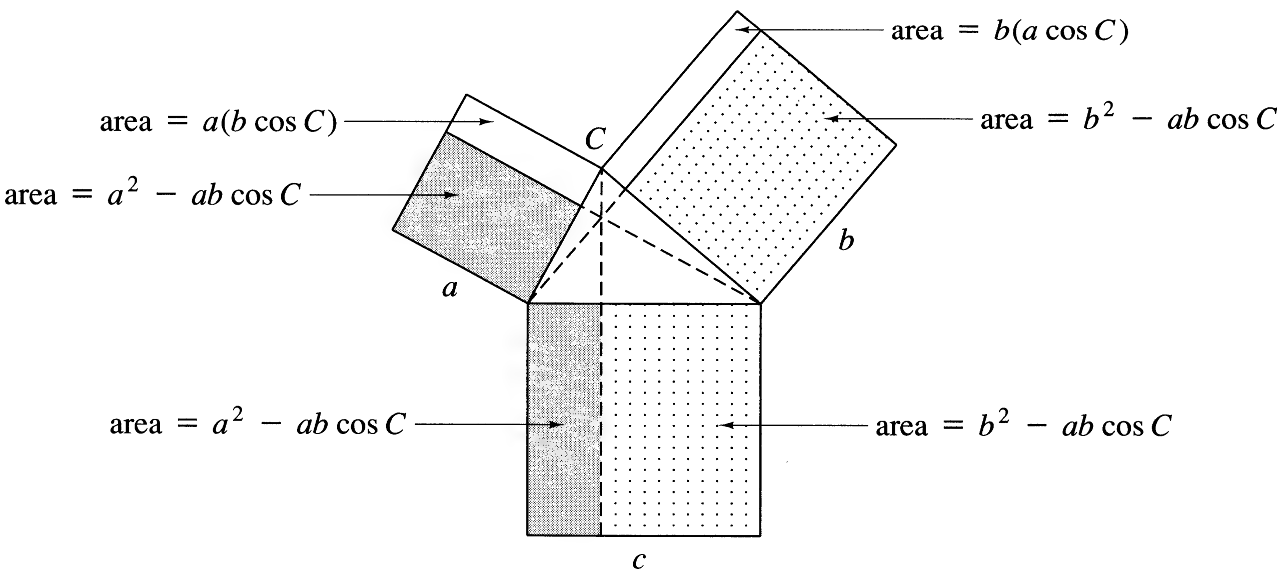


Figure 11. The correspondingly shaded rectangles have equal area.

dotted rectangle has area $b^2 - ab\cos C$. The shaded and dotted rectangles together fill out the square of side c, so we have

$$c^2 = (a^2 - ab\cos C) + (b^2 - ab\cos C),$$

or

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

which is the law of cosines.

In the foregoing argument angle C is smaller than a right angle. If C is larger than a right angle the two altitudes from A and B fall outside the opposite squares, and the shaded and dotted rectangles in Figure 11 extend outside the smaller squares of areas a^2 and b^2. In this case c^2 is greater than $a^2 + b^2$ by an amount equal to the areas of the two protruding rectangles. A modification of the foregoing argument shows that each of the protruding rectangles has area $ab\cos(\pi - C)$. But $\cos(\pi - C)$ is the negative of $\cos C$, a positive number when C is larger than a right angle, so the area of the two protruding rectangles is $-2ab\cos C$ (a positive number when $\cos C$ is negative) and we end up with the same formula for the law of cosines.

Another animation argument

There is a variation of the foregoing proof that does not require rotation. Refer to Figure 12, where angle C is smaller than a right angle. If the two squares with common vertex C are sheared as shown in Figure 12 they are transformed to two rectangles that overlap the square opposite angle C. It can be shown that each overlapping portion is a rectangle of area $ab\cos C$. Therefore the area c^2 is smaller than the sum $a^2 + b^2$ by the amount $2ab\cos C$, which is another way of stating the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos C.$$

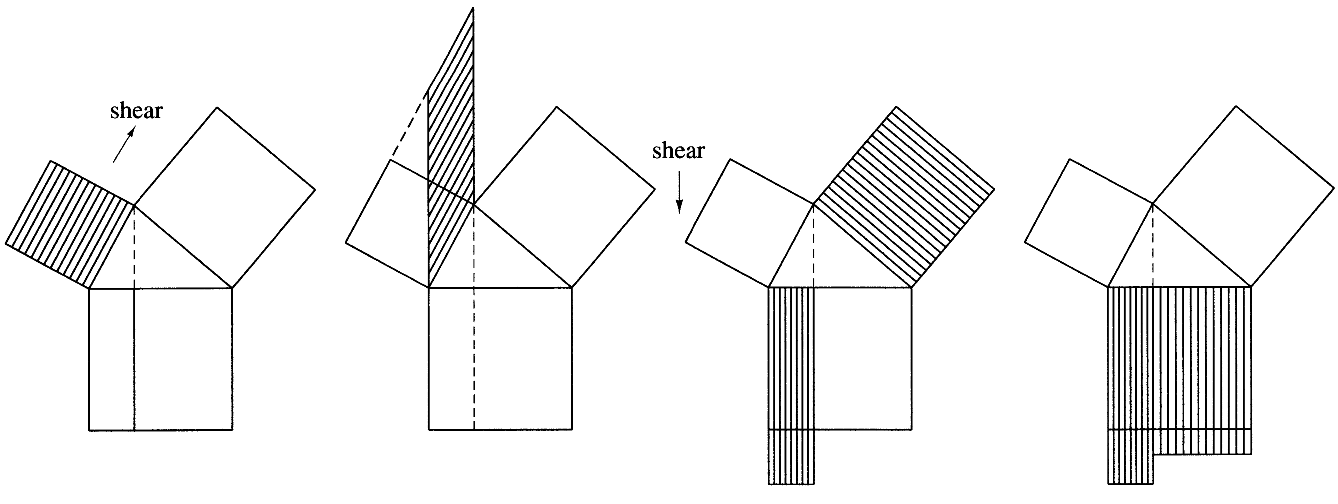
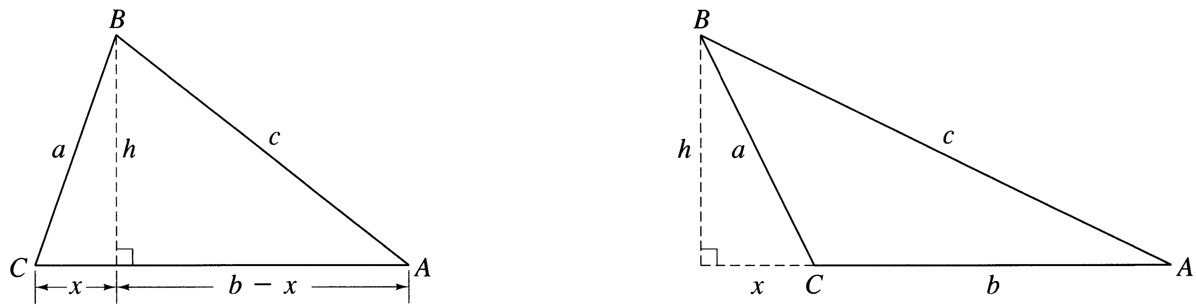


Figure 12. Animated proof using shearing but not rotation.

A proof based on the Pythagorean Theorem

Triangle ABC in Figure 13a has three acute angles. The altitude of length h from vertex B divides the triangle into two right triangles. The side of length b (opposite angle B) is divided into two segments whose lengths are x and $b - x$, as shown in Figure 13a. If angle C is greater than a right angle, the altitude from B to the opposite side falls outside the triangle, as shown in Figure 13b.



(a) Angle C smaller than a right angle. (b) Angle C greater than a right angle.

Figure 13. The law of cosines deduced from the Theorem of Pythagoras.

Now let's calculate h^2 in two ways, using the Theorem of Pythagoras on each of the two right triangles in Figure 13a. From the triangle on the left we find

$$h^2 = a^2 - x^2,$$

and from the other right triangle we find

$$h^2 = c^2 - (b - x)^2 = c^2 - b^2 + 2bx - x^2.$$

Equate the two expressions for h^2 and solve for c^2 to obtain

$$c^2 = a^2 + b^2 - 2bx.$$

But $x/a = \cos C$, hence $x = a \cos \mathrm{C}$ and the last equation becomes the law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

If angle C is greater than a right angle, the altitude from B to the opposite side falls outside the original triangle as shown in Figure 13b. Again we apply the Theorem of Pythagoras twice, once to the right triangle with hypotenuse a and legs x and h, and again to the right triangle with hypotenuse c and legs $x + b$ and h. Equating the formulas for h^2 we find

$$c^2 = a^2 + b^2 + 2bx.$$

This gives the law of cosines because, in this case, $x = a \cos(\pi - C) = -\, a \cos C$, so $2bx = -\, 2ab \cos C$.

4. *Applying the law of cosines*

Although we proved the law of cosines in the form

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

there are corresponding versions for determining the other two sides:

$$a^2 = b^2 + c^2 - 2bc\cos A,$$

and

$$b^2 = a^2 + c^2 - 2ac\cos B.$$

These equations tell us how to find the length of any side of a triangle in terms of the lengths of the other two sides and the cosine of the angle between them. In actual practice, the angles are measured with a protractor, an engineer's transit, or other instrument, and their cosines are found in tables or by using a calculator with a cosine key.

The law of cosines can also be used to find the angles of a triangle when all three sides are known. All we need do is solve each of the foregoing equations for the cosine of the angle in terms of the lengths of the sides. Thus, for example, the first equation can be written as follows:

$$2ab\cos C = a^2 + b^2 - c^2,$$

from which we find

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

And, of course, there are corresponding equations for the cosines of the other two angles.

If the cosine of an angle is known, the angle itself can be recovered by reading a table of cosines in reverse, or by using a calculator with a $\cos^{-1}$ key. If the cosine is positive the angle is smaller than a right angle, and if the cosine is negative it is greater than a right angle. Calculators have special settings (often called *modes*) that indicate whether angles are expressed in radians or in degrees.

Note that the quotient

$$\frac{a^2 + b^2 - c^2}{2ab}$$

is meaningful for any choice of positive numbers a, b, c. But the cosine of an angle is a number lying between -1 and $+1$. So, if the value of the quotient is outside this interval, there is no triangle having the edges a, b, c. For example, there is no triangle with edges $a = 1$, $b = 2$ and $c = 4$ because the quotient has the value $-11/4$.

The foregoing example shows that lengths of the sides of a triangle cannot be chosen arbitrarily. In fact, in any triangle the length of any side is always less than the sum of the lengths of the other two sides. This statement is called *the triangle inequality* and it is based on the intuitive idea that the shortest path between two points is the line segment joining them. Thus, for example, there is no triangle with edges 1, 2 and 4 because 4 is greater than the sum of 1 and 2. More generally, there is no triangle with edges a, b and c if $c > a + b$. In fact, it is easy to show that in this case the above quotient has a value less than -1 so it cannot be the cosine of any angle. (See Exercise 4, p. 16.)

The law of cosines can also be used to find one side of a triangle if we know the other two sides and the angle opposite one of them. For example, suppose we know a, b and angle A (opposite the side of length a). We can determine c from the equation

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

This is a quadratic equation for c that can be solved by the quadratic formula. In general the quadratic has two roots given by

$$c = b \cos A \pm \sqrt{(b \cos A)^2 - (b^2 - a^2)}\,.$$

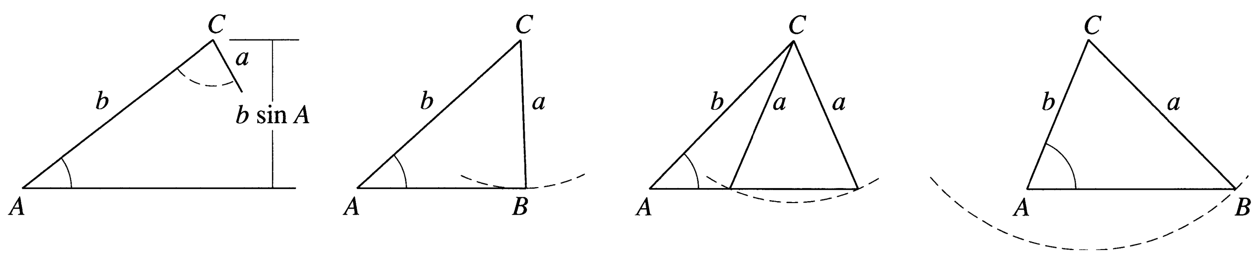
Because of the Pythagorean identity, $\cos^2 A$ can be replaced by $1 - \sin^2 A$ under the square root sign and the formula for c becomes

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}\,.$$

The quantity under the square root sign can be negative, zero, or positive, which means that the number of roots c can be zero, one, or two. For this reason, the problem of determining side c when sides a, b and angle A are given is called the ambiguous case. When A is smaller than a right angle ($A < 90°$), the various possibilities are shown in Figure 14. The problem can be analyzed as follows:

If $a < b \sin A$ the expression under the square root is negative and there is no real solution for c, hence no triangle having given sides a, b and angle A. This case is illustrated in Figure 14a. If $a = b \sin A$ the quantity under the square root is zero and there is exactly one solution for c, namely $c = b \cos A$, hence exactly one triangle with sides a, b and angle A, as shown in Figure 14b. If $b \sin A < a$ the quantity under the square root is positive and there are two solutions for c. If $a < b$ both values of c are positive and there are two triangles with given sides a, b and angle A, as illustrated in Figure 14c. But if $a \geq b$ one value of c is positive and the other is negative, so there is only one triangle with given sides a, b and angle A, as shown in Figure 14d. If $A \geq 90°$ the ambiguous case is discussed in Exercise 5, p. 16.

In the next section we will learn another method for treating the ambiguous case without the need to solve a quadratic equation. Instead of the law of cosines it uses another property called *the law of sines*.



$a < b \sin A$ (a) $a = b \sin A$ (b) $b \sin A < a < b$ (c) $a \geq b$ (d)

Figure 14. Various possibilities in the ambiguous case when A is smaller than a right angle.

Exercises using the law of cosines

1. Two points A and B are separated by a pond, and we wish to determine the distance AB. Both points A and B are accessible from a third point C such that $AC = 30$ meters and $CB = 40$ meters. Determine distance AB if angle ACB is (a) 90°; (b) 45°; (c) 120°.

2. A ladder 52 ft long is set 20 ft in front of an inclined buttress, and reaches 46 feet up along its face. Find the angle of inclination (greater than 90°) between the face of the buttress and the horizontal.

3. A city lot forms a quadrilateral $ABCD$ with a right angle at vertex B and an angle greater than 90° at the opposite vertex D. The lengths of its edges in feet are $AB = 423$, $BC = 162$, $CD = 420$, and $AD = 160$. Determine the angle in degrees at vertex D.

4. If three positive numbers a, b, c are such that $c > a + b$ there is no triangle with edges a, b, c. Show that in this case the quotient

$$\frac{a^2 + b^2 - c^2}{2ab}$$

is less than -1 (and, therefore, cannot be the cosine of any angle).

5. If two edges of a triangle with lengths a, b are given and if the angle A opposite the side of length a is given, the law of cosines becomes a quadratic equation for c:

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

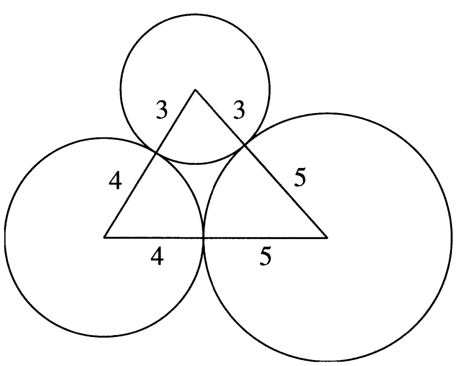
(a) Show that the roots of this quadratic can be expressed in the following form:

$$c = b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}\ .$$

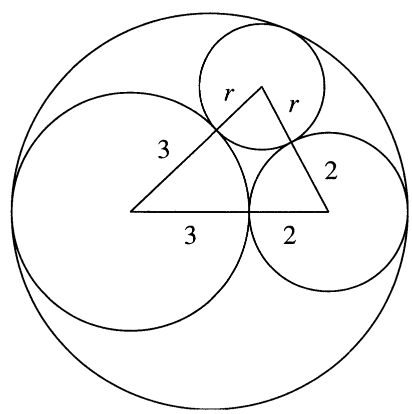
(b) If $A \geq 90°$, show that there is no triangle with side c if $a \leq b$, and exactly one triangle if $a > b$.

6. Three circles of radii 3, 4 and 5 are externally tangent. Show that the triangle with vertices at the three centers has angles with cosines 2/3, 2/7, and 11/21.

7. The diameters of two circles of radii 3 and 2 form the diameter of a larger circle of radius 5. Find the radius r of a fourth circle inside the large circle and tangent to all three given circles.

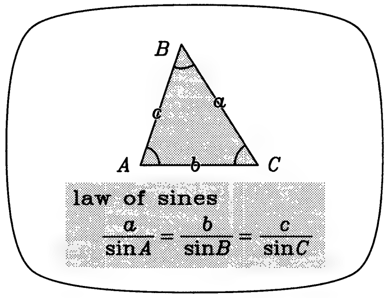


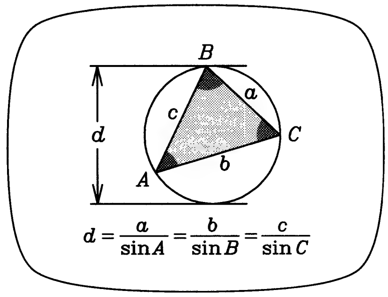
Exercise 6.



Exercise 7.

5. *The law of sines*





The law of cosines was derived by calculating the square of the altitude of a triangle in two ways. The same diagram leads to another important property of triangles called the *law of sines*. It says that in *every* triangle the ratio of the length of a side to the sine of the opposite angle is constant. To see why, refer to the triangle in Figure 15a. The sines of angles A and B are given by the following ratios:

$$\sin A = \frac{h}{b}, \quad \sin B = \frac{h}{a}.$$

We solve each of these equations for h and equate the results to obtain

$$b \sin A = a \sin B.$$

This can be written as follows:

$$\frac{a}{\sin A} = \frac{b}{\sin B}.$$

Figure 15. Law of sines: In any triangle the ratio $\frac{\text{length of side}}{\text{sine of opposite angle}}$ is the same for all three sides.

Repeating the argument with the altitude from vertex B (see Figure 15b) we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}.$$

Combining this with the previous equation we find

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad \textbf{(law of sines)}$$

This shows that the ratio of the length of a side to the sine of the opposite angle is the same for all three sides of the triangle, so in a given triangle this ratio is constant.

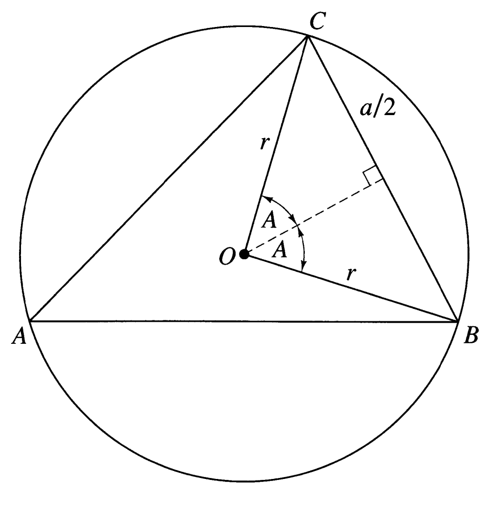
Incidentally, this constant ratio has an interesting geometric meaning. It is equal to the diameter of the circle passing through the three vertices of the triangle. To see why, refer to Figure 16a, which shows two radii of length r from the center O of the circle to the vertices B and C, forming a central angle that cuts off the same arc as the angle from vertex A. It is known that the measure of the central angle is twice that of angle A. (This is proved below, using Figure 16b.) Therefore, a perpendicular from O to the side BC bisects the central angle and forms two congruent right triangles each with an angle A at the common vertex at O. Calculating sin A from one of these right triangles we find

$$\sin A = \frac{a/2}{r},$$

and hence

$$\frac{a}{\sin A} = 2r .$$

But $2r$ is the diameter, so the constant ratio in the law of sines is equal to the diameter of the circle through the three vertices of the triangle.



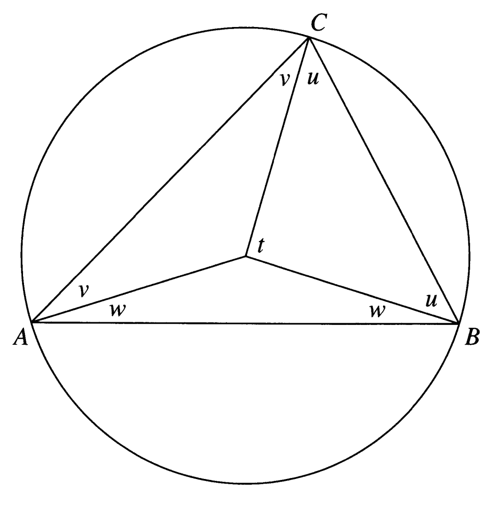


Figure 16. The constant ratio in the law of sines is equal to the diameter of the circle through the vertices.

The foregoing proof used the fact that the measure of the central angle O is twice that of inscribed angle A. To see why this is true, let t denote the measure of the central angle. We will show that $t = 2A$. The three radii to the vertices of triangle ABC form three isosceles triangles as shown in Figure 16b. Label their base angles u, u, v, v, and w, w. In the isosceles triangle with central angle t the sum of the angles is $2u + t$, and in triangle ABC the sum of the angles is $2u + 2v + 2w$. These two angle sums must be equal (because the sum of the angles in any triangle is a straight angle) so we have

$$2u + t = 2u + 2v + 2w,$$

which simplifies to

$$t = 2(v + w).$$

But $v + w = A$, so the last equation states that $t = 2A$, which is what we set out to prove.

Sines and chord lengths

In the argument above we showed that

$$\frac{a}{\sin A} = d,$$

the diameter of the circle through the vertices. When $d = 1$ (unit diameter) the equation becomes

$$\sin A = a.$$

This property is illustrated in Figure 17. It states that, *the length of a chord in a circle of unit diameter is equal to the sine of the subtended inscribed angle.*

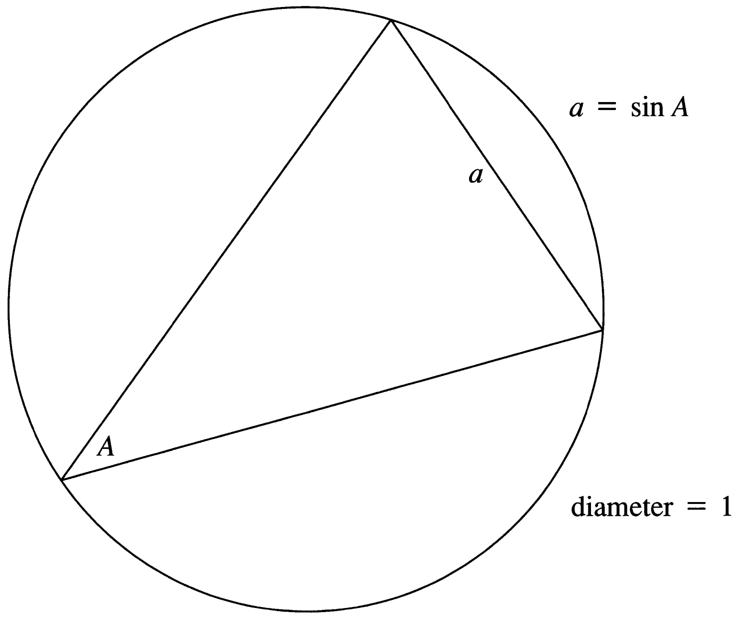


Figure 17. Chord length is equal to the sine of the subtended inscribed angle in a circle of unit diameter.

Because of this property, Ptolemy's theorems on chords in his *Almagest* (*c.* A.D. 150) translate directly into theorems on sines. One of these theorems, called the addition formula for sines, is described in a later program.

According to Theon of Alexandria (A.D. 390), Hipparchus (*c.* 130 B.C.) had written twelve books on the computation of chords of circles. These books are now lost, but because of their influence on Ptolemy and later writers, Hipparchus is often called the father of trigonometry. Because the *Almagest* survives, Ptolemy is also considered one of the founders of trigonometry. There's no doubt that Ptolemy learned much from Hipparchus' work.

Around 500 A.D., the Hindu mathematician Aryabhata was also computing half-chords. Somewhat later, tables of sines were computed by other Hindu mathematicians. The sine was called *jya*, one of several spellings of the Hindu word that means half-chord. Later the Arabs transliterated this to *jyb*, which was later incorrectly read as *jayb*, the Arab word for *pocket*, *gulf*, or *bosom* by the translator Gherardo of Cremona (*c.* 1150) who, in translating from Arabic to Latin, used the Latin equivalent *sinus*. This, in turn, became anglicized to *sine*.

6. *Applying the law of sines*

A triangle has six parts: three angles and three sides. If one side and two other parts are known the remaining parts can be found using the law of sines and/or the law of cosines, as shown by the following examples. As usual, the sides of lengths a, b, c are opposite the angles A, B, C, respectively.

***Example* 1. (One side and two angles known.)** From the two given angles we can find the third because the sum of all three angles is a straight angle. If side c is known the law of sines gives us the other two sides:

$$a = \frac{c}{\sin C}\sin A, \qquad b = \frac{c}{\sin C}\sin B.$$

The law of cosines can be used to check the calculations.

***Example* 2. (Two sides and the included angle known.)** If the given sides are a and b and the given angle is C, the law of cosines tells us that the positive square root of $a^2 + b^2 - 2ab\cos C$ is the third side c. Now we can determine $\sin A$ from the law of sines:

$$\sin A = \frac{a \sin C}{c}.$$

If $\sin A = 1$ then $A = 90°$. If $\sin A < 1$ we cannot determine A from a knowledge of $\sin A$ alone because $\sin A$ equals the sine of the supplement of A. However, both A and B cannot exceed 90°, and the smaller angle must lie opposite the smaller of the sides a and b. Therefore, if we label the sides so that $a \le b$ then we can be sure that A is smaller than a right angle. Angle A itself can be recovered from a table of sines or from a calculator with a $\sin^{-1}$ key. Angle B, of course, is the supplement of $A + C$.

***Example* 3. (Two sides and one opposite angle known: the ambiguous case.)** When this case is treated with the law of cosines it requires solving a quadratic equation. The analysis using the law of sines is simpler. Suppose the given sides are a and b and the given angle is A, where A is less than 90° as in Figure 18. If a triangle exists, the law of sines tells us that $\sin B$ is equal to the ratio $(b \sin A)/a$. The quantity in the numerator, $b \sin A$, represents the perpendicular distance from vertex C to the opposite side c. If the ratio $(b \sin A)/a$ is greater than 1, there is no solution (Figure 18a); if the ratio is equal to 1, angle B is a right angle and there is exactly one solution (Figure 18b); and if the ratio is less than 1 there are two solutions if $a < b$ (Figure 18c) and exactly one solution if $a \ge b$ (Figure 18d).

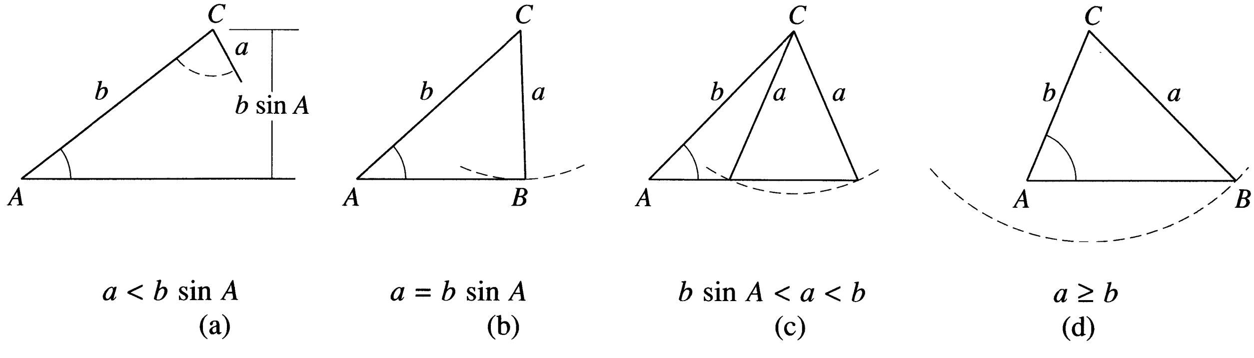
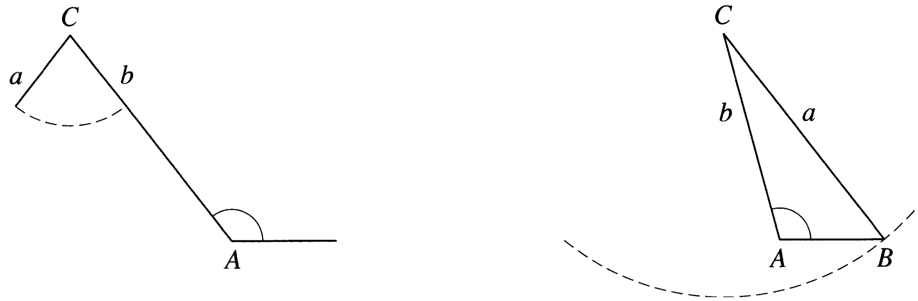


Figure 18. Various possibilities in the ambiguous case when $A < 90°$.

If $A \geq 90°$ there is no solution if $a \leq b$ (Figure 19a) and exactly one solution if $a > b$ (Figure 19b).



$a \leq b$
(a)

$a > b$
(b)

Figure 19. Two possibilities in the ambiguous case when $A \geq 90°$.

When a solution exists, as soon as we know two angles and one side the remaining parts can be found as in Example 1.

***Example* 4.** (**Three sides known.**) Label the sides so that $a \leq b \leq c$. If we solve first for the two angles opposite the smaller sides we can be sure they are smaller than 90°. The law of cosines gives us

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} .$$

Since a^2, b^2 and c^2 have already been computed we could use the law of cosines to determine $\cos B$, or we could use the law of sines to find $\sin B = (b \sin A)/a$. Both A and B are less than 90°, and C is the supplement of $A + B$.

Exercises using the law of sines

1. Two points A and B are separated by a pond, and we wish to determine the distance AB. Both points A and B are accessible from a third point C such that $AC = 40$ meters and angle BAC is 45°. Determine distance AB if angle ABC is (a) 45°. (b) 60°. (c) 90°.

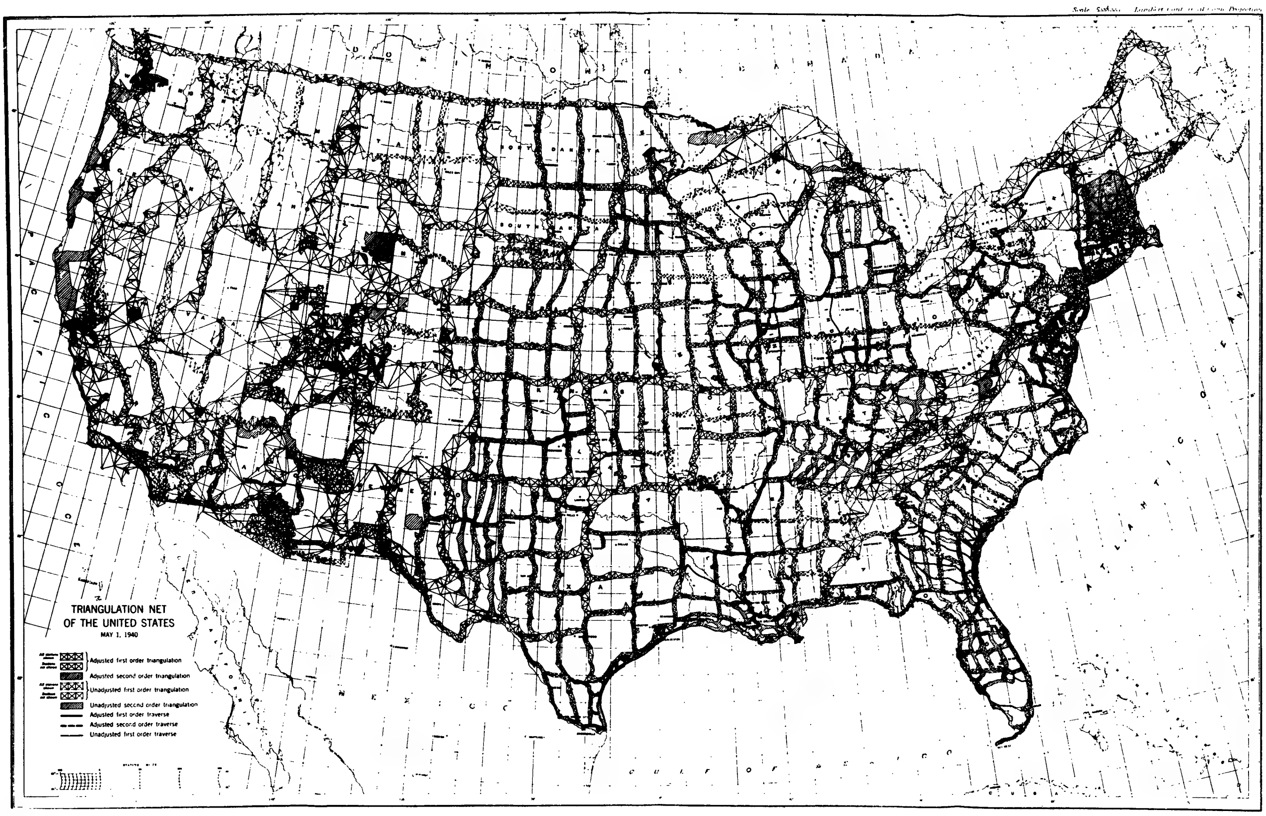
2. From a point on a horizontal plane the angle of elevation to the top of a tall monument is 40°, and 50 ft farther away in the same vertical plane the angle is 28°. How tall is the monument?

3. Refer to Exercise 3 on page 16, and determine the angles in degrees at vertices A and C.

4. A ship sails 12 miles from a lighthouse at point A on a bearing of S. 45° W. (that is, on a line measured from A to the west of south by an angle of 45°). From there it sails 15 miles on a bearing of S. 50° E. to point C. (a) Find the distance AC from the lighthouse to the ship. (b) Find the bearing for a ship to sail directly from A to C.

5. In each case, find the angles of a triangle with the following sides, or explain why no such triangle exists. (a) $a = 20$, $b = 99$, $c = 101$. (b) $a = 20$, $b = 99$, $c = 111$. (c) $a = 20$, $b = 99$, $c = 121$.

7. *Surveying by triangulation*



Modern geodetic surveying acquires information for determining the geographic positions of widely separated points on the earth's surface so that accurate maps may be constructed. A method called *surveying by triangulation* locates points called *stations* at the vertices of triangles. In surveying large geographic regions such as the continental United States or the subcontinent of India, thousands of triangles are joined together to form a triangulation system. Extensive triangulation systems have been constructed in almost every civilized country on earth.

The lengths and directions of one or more lines, called *base lines*, are determined by direct measurements and astronomical observations; and the lengths and directions of other lines in the triangles are calculated by measuring the angles between the various lines and applying trigonometric principles, usually the law of sines or the law of cosines. (Figure 20a.) The angles are measured with a precise instrument, such as a *theodolite*, which is part telescope and part protractor. Because a geodetic survey involves hundreds or sometimes thousands of triangles, measurements of angles and base line distances must be made with great care. To improve accuracy, overlapping triangles are used to form a belt of quadrilaterals as shown in Figure 20b. These belts are apparent in the triangulation net of the United States shown at the top of this page. As might be expected, most large scale surveys are carried out by official government organizations that establish standards for limits of accuracy.

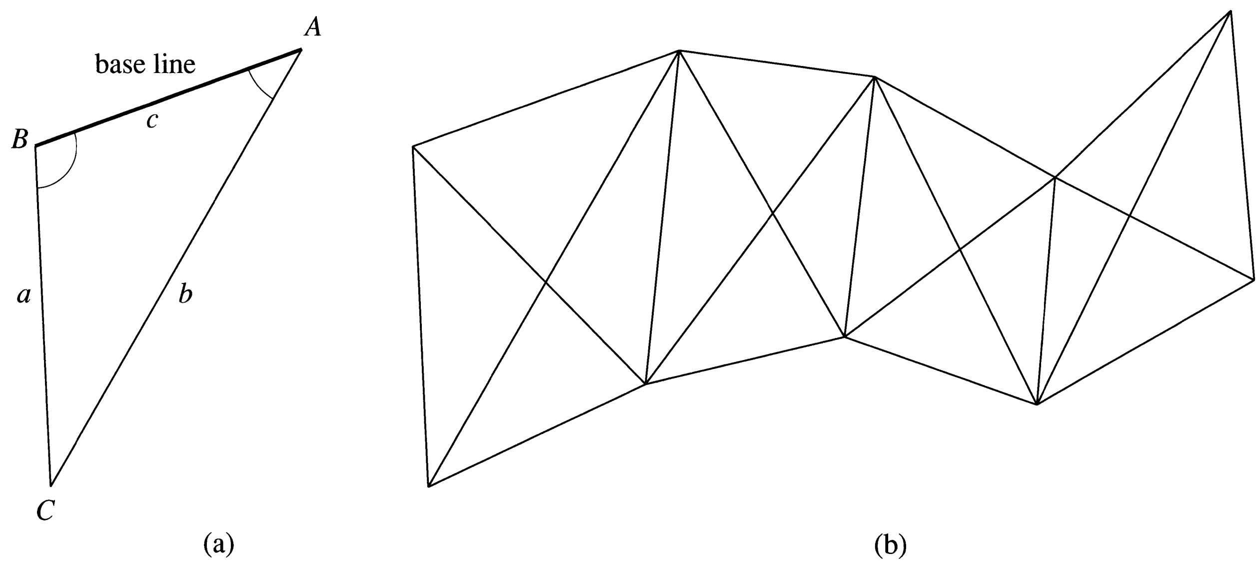


Figure 20. (a) The law of sines used in surveying by triangulation. (b) A belt of quadrilaterals.

In Figure 20a, the length of the base line, c, is established by direct measurement. The angles A and B are measured with a theodolite. Angle C is the supplement of $A + B$. The sides a and b are then calculated from the law of sines:

$$a = \frac{c}{\sin C} \sin A, \quad b = \frac{c}{\sin C} \sin B.$$

The law of cosines can be used to check the calculations.

The geographic position of a station is indicated by three numbers: (1) its *latitude* north or south of the equator; (2) its *longitude* east or west of a zero meridian (a great circle on the earth's surface passing through the north and south poles which, by agreement for most of the world, is the meridian through Greenwich, England); and (3) its *elevation* above mean sea level. In the United States, the actual mean level of the Atlantic and Pacific Oceans and the Gulf of Mexico has been determined over a period of years by measurements made at various bench marks along the coasts. An average elevation of the sea is established by the U. S. Coast and Geodetic Survey and is taken as mean sea level.

The base lines and the theodolite are used to determine latitude and longitude of the stations in the triangulation system. Altitudes above sea level are determined by a different process called *spirit leveling*. It employs a leveling instrument consisting of a telescope fitted with a spirit level, a tube nearly full of alcohol, ether, or a mixture of both. The position of a bubble in the tube indicates whether the telescope is horizontal. An engraved rod is placed vertically at a known elevation. A second vertical rod is then placed no more than 50 feet away. Each rod is observed through the telescope and the difference in elevation between them is found by subtraction. (Figure 21.) The first rod and the leveling instrument are then moved forward and the process is repeated thousands of times, thereby establishing the elevation of points along the survey route. In terrain that changes elevation gradually the process of spirit leveling is very accurate. But in steep, mountainous terrain, such as the Rocky Mountains or the Himalayas, vertical triangulation with the theodolite is used instead.

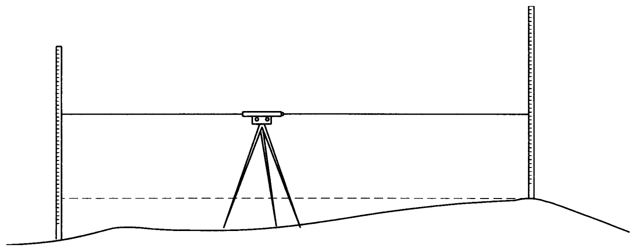


Figure 21. Change in elevation determined by spirit leveling.

In recent years, electronic equipment has been introduced in surveying. The Global Positioning System employs several orbiting satellites in fixed orbits about 12,500 miles high. The satellites transmit radio signals to receivers on the ground whose positions are taken as unknown. By measuring the time of arrival of the signals from at least three satellites simultaneously, the relative distances and altitudes of the receivers can be determined within a few seconds using a portable microwave receiver and computer. The receivers can then be used as stations in a traditional triangulation survey.

Brief history of surveying

Primitive methods of surveying can be traced back to the ancient Egyptians who measured the land with ropes and rods. The earliest extant book on surveying was written by Hero of Alexandria, an outstanding practical Greek scientist who flourished around 100 A.D. His *Treatise on the Dioptra* enunciates the first principles of surveying and engineering. The dioptra was an instrument of earlier Greek origin used for leveling and for measuring right angles. Vitruvius, who served as a military engineer under Julius Caesar, mentions the dioptra as an instrument used by Roman surveyors. Ancient surveying reached its peak of perfection with the Romans, who improved the instruments of the Egyptians and the Greeks. After the fall of Rome, many of these instruments were further improved during the fourteenth and fifteenth centuries by European instrument makers and mathematical practitioners, especially after the development of the telescope and the magnetic compass.

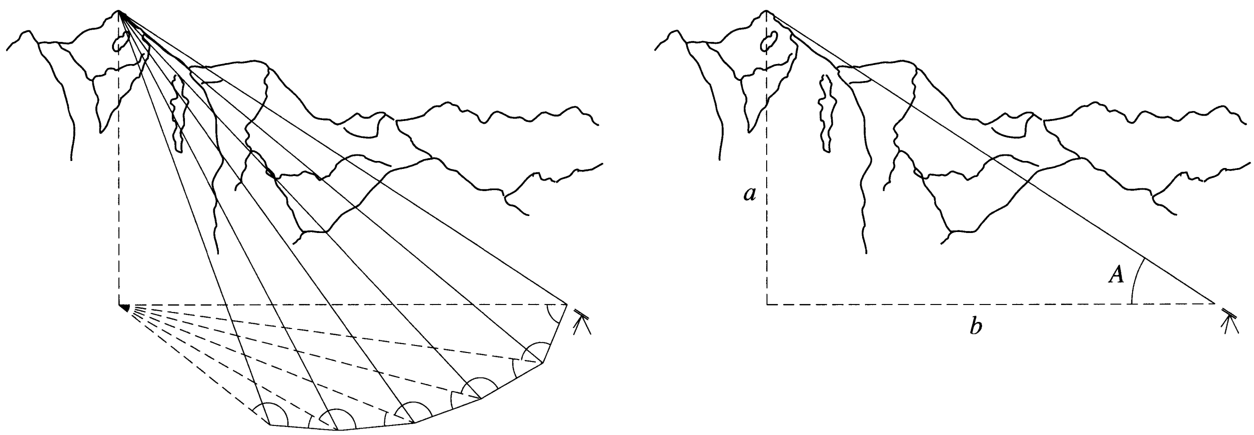
Surveying by triangulation was first described in print by Gemma Frisius (1508-1555) in his work, the *Libellus*, the first edition of which was bound with the 1529 Flemish edition of *Cosmographia*, a famous collection of maps produced by Petrus Apianus shortly after Columbus discovered the New World. The Libellus had a far-reaching effect on new maps that were prepared to satisfy the increasing demands of expanding empires that grew out of the great voyages of discovery.

The Great Trigonometric Survey of India

After 1750 many European governments undertook systematic mapping of their lands. When the British empire grew to include the Indian subcontinent, a young navy lieutenant, James Rennell, initiated the Survey of India, one of the great geographical efforts of all time. His *Bengal Atlas*, published in 1777, mapped the coastline and southern provinces of the Indian subcontinent. But the vast interior and the mountains to the north remained a complete mystery. In 1801, William Lambton began pushing the survey a thousand miles northward into the heart of the subcontinent. Exhausted by his 17-year effort, Lambton died in his tent. He was succeeded by George Everest, a young and ambitious captain who

undertook the task with a fanatic devotion. Everest directed an expedition of surveyors with an entourage of bearers and servants through difficult terrain abounding with poisonous snakes, tigers, and the deadliest enemy of all, the malaria bearing mosquito. Twenty five years later, with the snow-capped peaks of the Himalayas within view and the "great arc of India" completed, Everest's failing health forced him to retire to England. His successor, Andrew Waugh, extended the survey across northern India to the borders of Nepal, which were closed to foreigners. In November 1849 Captain James Nicholson, a member of the expedition, trained his theodolite on a distant summit from six locations on the Indian plain (Figure 22a.) The theodolite measured horizontal angles, and the law of sines gave the horizontal distance from each station to an inaccessible point directly below the summit. The vertical angle from each station to the summit was measured by the theodolite, and the common altitude a of each right triangle (Figure 22b) was calculated by the law of sines. When base b and elevation angle A are known the law of sines gives

$$a = b\frac{\sin A}{\sin(\frac{\pi}{2} - A)} = b\ \frac{\sin A}{\cos A}\ .$$



(a) (b)

Figure 22. Determining the height of Mt. Everest by trigonometry.

The average of the six calculations placed the height of the summit at 29,002 feet, the highest mountain on earth. It was renamed in honor of George Everest, who played such a major role in the great trigonometric survey illustrated in Figure 23.

With the opening of the Himalayas in the 1920s from Tibet and in the 1950s from Nepal, surveyors with direct access to the base of Mt. Everest remeasured the height at 29,028 feet, only 26 feet higher than the original calculations made a century earlier. Despite brutal weather and problems associated with excessively high altitude, many expeditions attempted to climb to the summit of Mt. Everest. Seven successive attempts from the northeast failed during the period 1921-1938, as did three attempts from the south in 1951-52. Finally, on May 29, 1953, Edmund P. Hillary from New Zealand and Tenzing Norgay from Nepal reached the top of the world and achieved international acclaim.

The Indian survey expedition measured the height of Mt. Everest by approaching it with a network of triangles from the south. A Chinese expedition made an independent survey from the north, starting from the Gulf of Bohai, thousands of miles to the east, and arrived at the same height for Everest, to within a few feet. The use of satellites has shown that the traditional way to measure the height of a mountain such as Mt. Everest, using leveling and theodolites, is highly accurate. But the story is not over. Because of continuing movement of the earth's surface, the height of Mt. Everest could be changing with time.

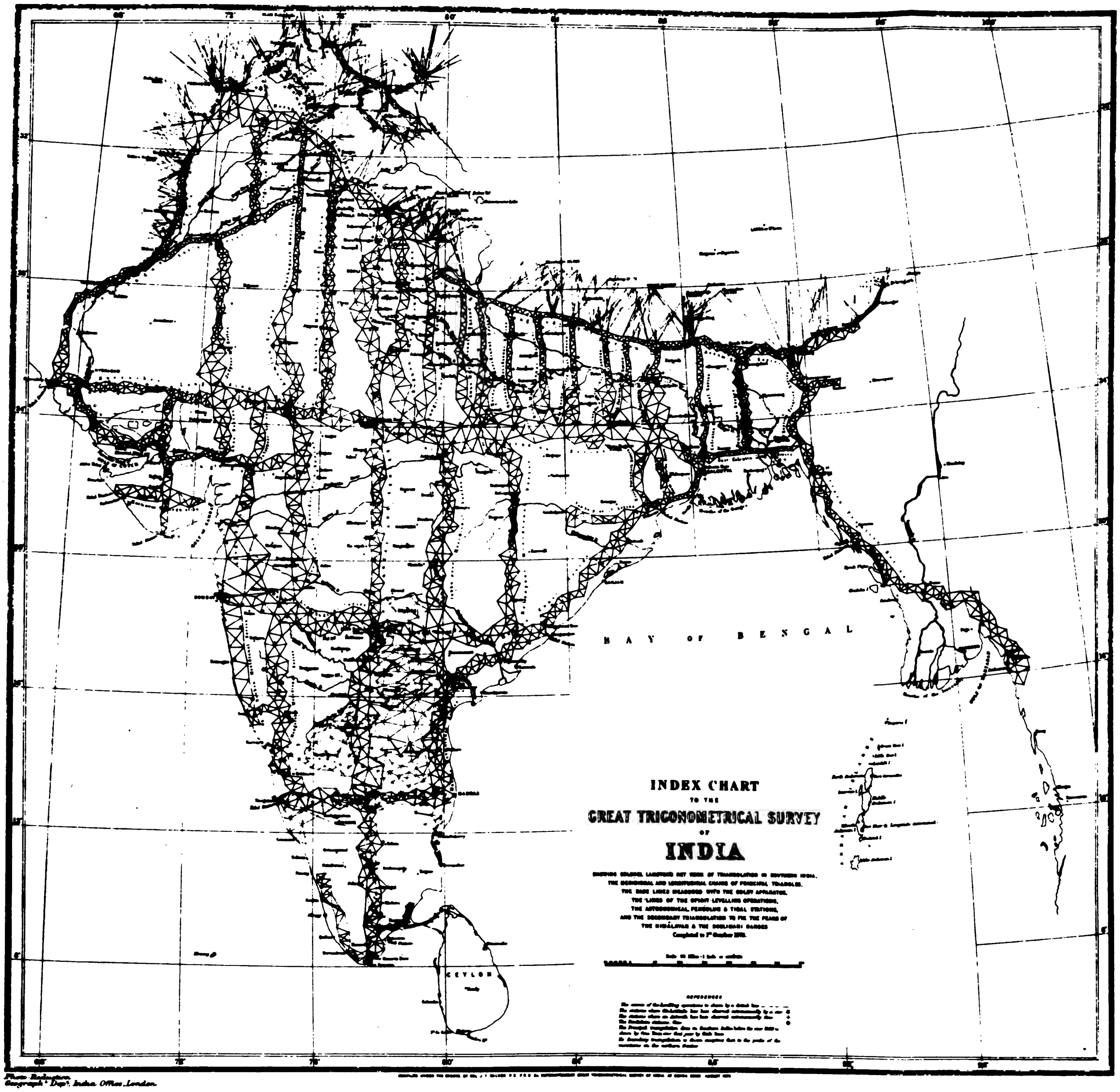


Figure 23. Triangulation network produced by the Great Trigonometrical Survey of India.

8. *Recap of Parts I and II. Preview of Parts III and IV.*

Sines and Cosines, Part I. This module shows how sines and cosines arise in different contexts. They occur as the rectangular coordinates of a point moving on a unit circle and, as such, they are periodic functions (called the circular functions). Sines and cosines are used to analyze musical sounds and other periodic vibrating phenomena. Sines first appeared in ancient astronomy as lengths of chords of circles. Sines and cosines were studied later as ratios of lengths of sides of right triangles.

Sines and Cosines, Part II. The principal focus of this segment is the *law of cosines* and the *law of sines*, both of which are useful in determining sides and angles of a triangle when three elements, at least one of which is a side, are known. The law of cosines is an extension of the Pythagorean theorem

for right triangles. The law of sines states that in any triangle the ratio of the length of a side to the sine of the opposite angle is constant, this constant being the diameter of the circle through the three vertices of the triangle.

The law of sines is especially useful in surveying by triangulation. Primitive methods of surveying go back to the ancient Egyptians and the Greeks. The earliest extant book on the principles and instruments of surveying was written by Hero of Alexandria around 100 A.D. The methods were improved by the Romans as they expanded their empire. Early Roman maps of relative small regions, such as cities, were remarkably good, considering the instruments available in Roman times. But their large scale maps of the known world were often greatly distorted and certainly not accurate by modern standards. Accurate surveying of large geographic regions occurred after Gemma Frisius introduced the method of surveying by triangulation in 1529. This method, together with improved instruments incorporating the telescope and the magnetic compass, resulted in new maps that followed the great voyages of discovery.

The first large scale survey by triangulation was undertaken during the eighteenth and nineteenth centuries. This was the great trigonometric survey of India that utilized five different expeditions and required more than a hundred years to complete. This survey also provided the first accurate measurement of the height of the world's tallest peak, Mt. Everest, named after one of the directors of the trigonometric survey of India.

Sines and Cosines, Part III. The principal focus of Part III is the *addition formulas* for the sine and cosine, which tell us how to determine the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles themselves. These formulas are part of a long list of trigonometric identities that are listed at the end of this section for easy reference. Some of these identities make it possible to express sines and cosines of many angles in terms of square roots of integers. The addition formulas also show that a linear combination of a sine and cosine with the same frequency is a shifted sine or a shifted cosine; this plays an important role in the study of simple harmonic motion.

Sines and Cosines, Part IV. When sines and cosines are calculated as ratios of sides of triangles, the accuracy of the final result may be limited by the degree of precision in measuring the lengths of sides. Part IV shows how to calculate sines and cosines to any desired degree of accuracy using polynomial approximations. These approximations are often used in designing hand calculators. When you press a sine or cosine key, the program inside the calculator often computes a polynomial approximation that gives the accuracy required.

Part IV also describes relations connecting areas and slopes associated with sine and cosine curves. The area of the region under a cosine curve and above the interval from 0 to t is $\sin t$, and the area of the region under a sine curve over the same interval is $1 - \cos t$. The slope of a sine curve at an arbitrary point $(t, \sin t)$ is $\cos t$, whereas the slope of a cosine curve at $(t, \cos t)$ is $-\sin t$. In establishing these relations we get a glimpse of the world of integral and differential calculus.

Wheels and machines executing repetitive circular motion dominate our way of life. Periodic phenomena are present throughout the universe, from the atomic scale to the solar system and beyond. Everything we see or hear comes to us in the form of wave motion. The ability to analyze all these phenomena using sines and cosines is a major achievement in mathematics.

List of basic properties of sines and cosines

The basic properties of the sine and cosine functions are listed here (with descriptive titles) for easy reference. Properties (b) through (j) are called *trigonometric identities* because they are valid for all values of x and y. The identities in (h), (i) and (j) can be deduced from properties (a) through (g).

(a) **Special values:** $\sin 0 = 0, \quad \sin\frac{\pi}{2} = 1, \quad \sin\pi = 0, \quad \sin\frac{3\pi}{2} = -1.$

$\cos 0 = 1, \quad \cos\frac{\pi}{2} = 0, \quad \cos\pi = -1, \quad \cos\frac{3\pi}{2} = 0.$

(b) **Periodicity:** $\sin(x + 2\pi) = \sin x$ and $\cos(x + 2\pi) = \cos x$.

(c) **The Pythagorean identity:** $\sin^2 x + \cos^2 x = 1$.

(d) **Odd property of the sine:** $\sin(-x) = -\sin x$.

(e) **Even property of the cosine:** $\cos(-x) = \cos x$.

(f) **Co-relations:** $\sin(\frac{\pi}{2} - x) = \cos x$ and $\cos(\frac{\pi}{2} - x) = \sin x$.

(g) **Addition formulas:**

$$\sin(x + y) = \sin x \cos y + \cos x \sin y .$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y .$$

(h) **Subtraction formulas:**

$$\sin(x - y) = \sin x \cos y - \cos x \sin y .$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y .$$

(i) **Duplication formulas:**

$$\sin 2x = 2 \sin x \cos x .$$

$$\cos 2x = \cos^2 x - \sin^2 x .$$

(j) **Difference formulas:**

$$\sin x - \sin y = 2 \sin\frac{x - y}{2} \cos\frac{x + y}{2} .$$

$$\cos x - \cos y = -2 \sin\frac{x - y}{2} \sin\frac{x + y}{2} .$$

In any triangle with angles A, B, C and opposite sides of lengths a, b, c the following relations hold:

(k) **The law of cosines:**

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

$$b^2 = a^2 + c^2 - 2ac \cos B,$$

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

(l) **The law of sines:**

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} .$$

The constant ratio in the law of sines is the diameter of the circle passing through the vertices of the triangle.

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